Some Propositions on Mathematical Models of Population Growth

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Abstract-In this study, we evoke the limitations of some well-known Ordinary Differential Equation models that apply on the growth/ decay rates of microbial populations. Hence, we propose an alternative perspective by implementing the Caputo's derivative of a function with respect to another function. Furthermore, new concepts: the growth of a population with respect to another population, and the infinite product of the rates of change of a pure culture population or mixed culture population are discussed.

Keywords – Population Growth, Fractional Calculus, Mittag-Leffler Function, Caputo-types Derivatives.

I. INTRODUCTION

Quantity rises over time through a process called exponential growth. It happens when the derivative, or instantaneous rate of change, of a quantity with respect to time is proportionate to the original quantity. When the proportionality constant is negative, the amount degrades with time and is referred to as experiencing exponential decay.

Mathematical modeling is a branch of mathematics that aids us in comprehending real-world problems, articulating them in mathematical models, and interpreting the results in the actual world. Mathematic modelling is the process of creating a mathematical model. It is a system interpretation based on a variety of concepts and packages. Mathematical models can be depicted in a number of different ways. Continuous ordinary differential equations (ODEs) or discrete difference equations are the most popular representations. Other examples include partial differential equations (PDEs), lattice gas automata (LGA), and cellular automata (CA), among others. The emphasize of this work is on models represented by ODEs, for example the following exponential population growth

$$\frac{dC(t)}{dt} = \alpha\, C(t) \qquad \rightarrow C(t) = C_0 e^{\alpha t} \qquad (1)$$

Where $C(t)$is the population as a function of time t,
$C_0 = C(0)$: the initial population size when t = 0,
α : growth (decay) constant derived from observations or data collection within a limited time interval.

Remark 1. Certainly, if we investigate the same population growth in another time interval, we might find different values of α. In this case, the solution in (1) might fail to be accurate especially in long term. Indeed, the exponential growth might not be realistic when time is sufficiently large.

In this work, we assume that is α itself might be a of function of time, in another words $\alpha = \alpha(t)$, which might be linear, polynomial, trigonometric or any type of functions . Here, we will only consider the case when α is an exponential function say $\alpha = e^{-\beta t}$.

II. FRACTIONAL CALCULUS

Fractional calculus is a branch of Mathematical analysis that explores all sort of powers for a given operator, such powers can be real or even complex numbers [1, 8, 10,11,15,20 & 21]. For the last five years the applications of fractional calculus have gained much attention in the fields of economics, physics, engineering and biology [11, 14, 18]. By swapping the order of the ordinary derivative with the fractional integral operator [9, 16], one can reconstruct the definition of Riemann–Liouville fractional derivative. By doing so, the Laplace transformation depends on the initial conditions of this new integer order derivative, in contrast to the initial conditions of fractional order derivative while using the Riemann–Liouville fractional derivative. In this article we consider an analogous concept namely the fractional derivative and the integral of a function with respect to another function via the following definitions [1].

Definition1. Let $\theta > 0$. $n \in N$. I is an interval $-\infty \leq a \leq b \leq \infty$. f is integrable function defined on I and $g \in C^1(I)$such that g is strictly increasing and $g' \neq 0$. for all $t \in I$. Then the fractional integral of function f with respect to another function g is given by

$$I_{a+}^{\theta.g} f(t) := \frac{1}{\Gamma(\theta)} \int_a^t g'(\tau)[g(t) - g(\tau)]^{\theta-1} f(\tau) d\tau. \quad (2)$$

Note that, if $g(t) = t$ the above assertion reduces to the Riemann-Liouville fractional integral.

By the means of definition 1, we can consider the Caputo's derivative of f with respect to g as follows:

Definition2. Let $\theta > 0$. $n = [\theta] \in N$. $I = (a.b)$ is an interval $-\infty \leq a \leq b \leq \infty$. $f \in C^n(I)$. $g \in C^1(I)$two functions such that g is strictly increasing and $g' \neq 0$. for all $t \in I$. The right fractional derivatives of f with respect to $g$are respectively given by

$$cD_{a+}^{\theta.g} f(t) := I_{a+}^{n-\theta.g} \left(\frac{1}{g'(t)} \frac{d}{dt} \right)^n f(t). \quad (3)$$

If $f(t) = [g(t) - g(a+)]^{\beta-1}$. $\beta > 1$ then:

$$cD_{a+}^{\theta.g} f(t) = \frac{\Gamma(\beta)}{\Gamma(\beta - \theta)} [g(t) - g(a+)]^{\beta-\theta-1} \quad (4)$$

We recall the Mittag-Leffler functions that have been considered by many authors due to their central role in the studies of fractional differential equations and their applications, one may refer to some recent monographs studied by [12, 13 & 14].

As Almeida recently mentioned in [2], the compositions of Mittag-Leffler functions with other functions can potentially be useful. The use of Integro-differential equations can provide solid theoretical underpinnings for these investigations. For that we provide the following:

If $f(t) = E_\theta \{\lambda [g(t) - g(a+)]^{\beta-\theta-1}\}$. $\lambda \in R$

$$cD_{a+}^{\theta \cdot g} f(t) = \lambda f(t) \quad (5)$$

III. PROPOSITIONS

A. FRACTIONALIZATION OF AN ODE

In view of remark 1 in the previous section, and letting $\alpha = e^{-\beta t}$ the ODE in (1) reduces to

$$e^{\beta t} \frac{dC(t)}{dt} = C(t)$$

We fractionalize the above by introducing the fractional derivative of $C(t)$with respect to the strictly increasing function $g(t) = e^{\beta t}$ and receive

$$cD_{a+}^{\theta\cdot g} C_\theta(t) = C_\theta(t). \quad \theta \in (0.1]. \text{ with initial condition } C_\theta(0) = C_0 \tag{6}$$

$$cD_{a+}^{\theta\cdot g}\ C_\theta(t) = \frac{1}{\Gamma(1-\theta)} \int_0^1 \left(\frac{e^{-\beta\tau} - e^{-\beta t}}{\beta}\right)^{-\theta} \frac{dC_\theta}{d\tau}\, d\tau. \tag{7}$$

This is a particular case of the Caputo-type fractional derivative of a function with respect to another function as we assumed that $g(t) = e^{\beta t}$ withthe restriction of $\theta \in (0.1]$. It is evident that Equation (6) is an Eigenvalue problem for the Caputo-type fractional operator defined in Equation (7), whose solution is given by

$$C_\theta(t) = C_0 E_\theta\left[\frac{1}{\beta^\theta}\left(1 - e^{-\beta t}\right)^\theta\right] \tag{8}$$

We compare the ODE and the fractional calculus solutions $C(t)\ Vs\ C_\theta(t)$

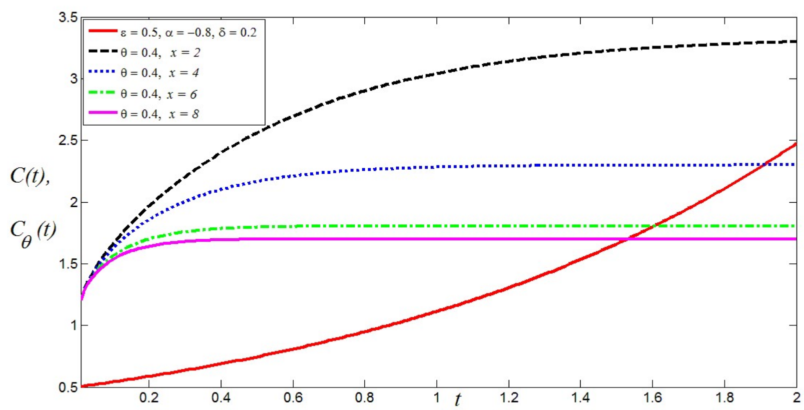

Figure1: C(t) vs Cɵ (t), C(t) in equation (1) for the values ε= 0.5, δ = 0.2, a = - 0.8 and Cɵ (t) in equation (8) for ϴ = 0.4 and four representative values of β =2 (Black), 4 (Blue), 6 (Green) and 8 (Magenta)

remark 2. from the graph, the ODE solution (in Red) is strictly increasing and growing faster, that may be considered as inaccurate for long run, especially when $t \to \infty$. However, all fractional calculus solution and regardless to parameters 'variation remain increasing but in an almost logarithmic way which might be more realistic.

Note that, if we let $\alpha = e^{\beta t}$, we will deduce the same conclusion for population decay.

B. The Growth of a population $C(t)$with respect to another decaying population, $N(t)$

In this section, we replace the growth (decay) rate of a population by another population, assuming that the two populations influence each other. We address the concept as follows:

Let $C(t), N(t)$ be two populations such that $N(t) = N_0 e^{-\beta t},\ \beta > 1$, if

$$\frac{dC(t)}{dt} = N(t)C(t) = N_0 e^{-\beta t}\, C(t)$$

Then the ODE model solves as:

$$C(t) = C_0 e^{\frac{N_0}{\beta}} \exp\left[-\frac{N_0}{\beta} e^{-\beta t}\right], \qquad C_0 = C(0),\ N_0 = N(0)$$

Remark3

i. $C(t)$ *is also a decaying population.*

ii. $t \to \infty \Rightarrow C(t) \to C_0 e^{\frac{N_0}{\beta}}$, *that is* $C(t)$ *will not vanish but decays* ***rapidly*** *to* $C_0 e^{\frac{N_0}{\beta}}$.

However, the fractional calculus model provides

$$e^{\beta t}\frac{dC_\theta(t)}{dt} = N_0 C_\theta(t)$$

$$C_\theta(t) = C_0 E_\theta\left[\frac{N_0}{\beta^\theta}\left(1 - e^{-\beta t}\right)^\theta\right]$$

Remark 4

i. $C(t)$ *is also a decaying population.*

ii. $t \to \infty \Rightarrow C(t) \to C_0 e^{\frac{N_0}{\beta}}$, *that is* $C(t)$ *will not vanish but decays* ***slowly*** *to* $C_0 E_\theta\left[\frac{N_0}{\beta^\theta}\right]$.

C. The infinite product of populations' rates of a growth

In this section, we explore the influence of the populations' growth(decay) for infinitely many populations. We consider the Logistic equation

$$\frac{dN}{dt} = \alpha N\left(1 - \frac{N}{K}\right), \alpha: growth\ rate, K: Carrying\ capacity$$

That solves as:

$$N = \frac{K}{1 + \left(\frac{K-N_0}{N_0}\right)e^{-rt}}, \quad t \to \infty \to N = K$$

Now, assuming an infinite interaction of rates of growth, i.e. if we evoke the following product in two cases:

1. Case 1: the infinite product of a single ODE

$$J(t) = \prod_{i=1}^{\infty}\frac{dN}{dt} = \prod_{i=1}^{\infty}\alpha\left(1 - \frac{N}{K}\right) = \alpha\prod_{i=1}^{\infty}\left(1 - \left(\sqrt{\frac{N}{K}}\right)^2\right)$$

We use the following identity

$$\frac{\sin \pi s}{\pi s} = \prod_{k=1}^{\infty}\left(1 - \frac{s^2}{k^2}\right)$$

Thus,

$$J(t) = \alpha \prod_{i=1}^{\infty}\left(1 - \left(\sqrt{\frac{N}{K}}\right)^2\right) \approx \alpha \frac{\sin \pi\left(\sqrt{\frac{N}{K}}\right)}{\pi\left(\sqrt{\frac{N}{K}}\right)}$$

Now, when $t \rightarrow \infty$ the population N reaches its carrying capacity K implying that

$$t \rightarrow \infty \Rightarrow J(t) = \alpha \prod_{i=1}^{\infty}\left(1 - \left(\sqrt{\frac{N}{K}}\right)^2\right) \approx \alpha \frac{\sin \pi\left(\sqrt{\frac{N}{K}}\right)}{\pi\left(\sqrt{\frac{N}{K}}\right)} = \alpha \frac{\sin \pi}{\pi} = 0$$

2. Case 2: The infinite product of infinity many ODEs

$$M(t) = \prod_{i=1}^{\infty}\frac{dN_i}{dt} = \prod_{i=1}^{\infty}\alpha_i\left(1 - \frac{N_i}{K_i}\right)$$

Now, when $t \rightarrow \infty$ each population N_i reaches its carrying capacity K_i implying that

$$t \rightarrow \infty \Rightarrow M(t) = \prod_{i=1}^{\infty}\frac{dN_i}{dt} = \prod_{i=1}^{\infty}\alpha_i\left(1 - \frac{N_i}{K_i}\right) = 0$$

Remark 5

i. $M(t) \rightarrow 0$, *whenever a population* N_i *reaches its carrying capacity* K_i *regardless to which population achieves its Maximum first.*

ii. $J(t) \rightarrow 0$, *whenever the population N reaches its carrying capacity K. However, since*

$$J(t) \approx \alpha \frac{\sin \pi\left(\sqrt{\frac{N}{K}}\right)}{\pi\left(\sqrt{\frac{N}{K}}\right)} \rightarrow J(t) = 0, whenever \sqrt{\frac{N}{K}} = n\ (+ve\ integer)\ that\ is\ N = n^2K, which\ is\ impossible$$

IV. CONCLUSION

The alternative fractional calculus models have been discussed in sections A, B providing an almost logarithmic growth (decay) behavior, that differs from the original ODEs models in the sense that ODE models grow (decay) rapidly, while fractional models grow (decay) slowly.

In section B, the influence of two populations on each other was evoked. However, the correspondent realistic and existing phenomena remain unknown subject to conducting some lab experiments in order to clarify the fact that there will be only three pathways of a population*C(t)* in the bioreactor it could be Decaying with time as the other population*N (t)* decays as well (Natural pathway), or *C (t)* will be an increasing population as a result of that *N (t)*raises a secondary metabolite that is *C(t)* may utilize as substrate and finally, *C(t)* remains as a steady population if *N(t)* secondary metabolite doesn't negatively effect on it.

For the conclusion of section C, a possible lab experiments involving multi species could be by the means of a suitable bioreactor type in case of studying pure and mixture bacterial growth cultivation is fed- batch bioreactor. Whereas in such kind of bioreactor the population growth of bacteria can be controlled by determine the cultivation period of time and an infinite growth of bacteria can be achieved.

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