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AN ANALYTICAL APPROACH TO UNDERSTAND PROJECTIVE, INJECTIVE AND FLAT MODULES

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Abstract- This article sets out a way to deal with know the idea of different modules in vector space as projective modules, injective modules, and flat modules. The definition and properties of the modules are clarified and talked about personally. During this specific circumstance, it's essential to comprehend that a vector space is furthermore called linear space. It's a progression of items, alleged vectors, which will be added and multiplied by numbers. During this specific circumstance, they're called scalars. A module is one of the elementary algebraic structures used in unique abstract algebra. A module on a ring is regularly speculation of the idea of vector space on a field.

Keywords: Vector space, Modules and their types as Projective modules, Injective modules and Flat modules.

I. INTRODUCTION

A vector space might be an assortment of components called vectors which will be added and multiplied by numbers, called scalars. Scalars are routinely viewed as real numbers, however there additionally are vector spaces with scalar increase by complex numbers, rational numbers, or for some fields. The undertakings of vector addition and scalar multiplication must meet certain necessities, which are referenced as axioms. to point that the scalars are real or complex numbers, the terms real vector space and complex vector space are consistently utilized.

Vector spaces are the subject of linear algebra are spoken to all around by their dimension, which for the most part decides the measure of autonomous bearings / independent directions in space. Vector spaces of infinite dimension ordinarily emerge in logical examinations like functional spaces, whose vectors are functions.

Let F be a field. A set V is called a vector space over F if there is an operation of addition on V , and a scalar multiplication function. From $F \times V$ to V , such that the following properties are satisfied.

i. Associative Law: $(u+v)+w=u+(v+w)$ forall $u,v,w \in V$.
ii. Commutative Law: $u+v=v+u$ for all $u,v \in V$.
iii. Existence of Identity: There exist an element $0 \in V$ such that $0+v=v+0$ for all $v \in V$.
iv. Existence of Inverse: For each $v \in V$ there exist $u \in V$ such that $u+v=v+u=0$.
v. Identity Element of Scalar Multiplication: $1.v$ For all $v \in V$.
vi. Compatibility of Scalar Multiplication with Field Multiplication: $(\alpha\beta)v=\alpha(\beta v)$ for all $\alpha,\beta \in F$ and $v \in V$.
vii. Distributivity of Scalar Multiplication with respect to Field Addition: $(\alpha+\beta)v=\alpha v+\beta v$ for all $\alpha,\beta \in F$ and $v \in V$.
viii. Distributivity of Scalar Multiplication with respect to Vector Addition: $\alpha(u+v)=\alpha u+\alpha v$ for all $\alpha \in F$ and $u,v \in V$.

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The meaning of a vector space likewise can be clarified in a few different ways. for example , the definition in certainty notes has two extra axioms: the total of two vectors must be a vector, and in this manner the numerous of a vector by a scalar might be a vector. These adages are a piece of the meaning of vector addition and scalar multiplication operations. Obviously, these are just two different ways to record a proportional definition: in the two cases, the total of two vectors must be a vector and along these lines the scalar multiplier of a vector with a scalar must be a vector. despite how it's composed.

Both the field and in this manner the vector space have zero components, the two of which are generally assigned with the sumbol. With a touch care, we will consistently tell from the context whether the zero scalar or the zero vector implies. Now and then we recognize vectors and scalars by documentation by composing a tilde under the vectors. would subsequently show the zero of the vector space being talked about.

II. SUBSPACE

A subset W of a vector space $0 \to A \to B \to P \to 0$ is called a subspace of V if W is itself a vector space under the addition and scalar multiplication defined on V .

In general, the ten axioms of the vector space must be checked to show that a set W with addition and scalar multiplication forms a vector space. However, if W it is part of a larger set V that is already known to be a vector space, some axioms do not need to be checked for W as they are inherited from V.

III. MODULES & SUB MODULES

A vector space M over a field R is a set of objects called vectors, which can be added, subtracted and multiplied by scalars (members of the underlying field). Thus M is an abelian group under addition, and for each $r \in R$ and $x \in M$ we have an element $rx \in M$. Scalar multiplication is distributive and associative, and the multiplicative identity of the field acts as an identity on vectors. Formally,

$$r(x+y) = rx + ry, (r+s)x = rx + sx, (rs)x = r(sx), 1.x = x.$$

For all $x, y \in M$ and, $s \in R$. A module is just a vector space over a ring. The formal definition is exactly as above, but we relax the requirement that R be a field, and instead allow an arbitrary ring. We have written the product rx with the scalar r on the left, and technically we get a left R-module over the ring R. The axioms of a right R-module are

$$(x+y)r = xr + yr, x(r+s) = xr + xs, x(rs) = (xr)s, x.1 = x.$$

If N is a nonempty subset of the R-module M , we say that N is a Sub-module of $M(N \le M)$ if for every $x, y \in N$ and $r, s \in R$, we have $(rx + sy) \in N$. If M is an R-algebra, we say that N is a sub-algebra if N is a sub-module that is also a subring.

IV. DEFINITIONS

Definition 4.1: (*R*-Homomorphism between two modules)- "Let M and N be two R-modules, then a function $f: M \to N$ is a homomorphism in case for all $a, b \in R$ and all $x, y \in M; f(ax+by) = af(x) + bf(y)$. The function f must preserve the defining structure in order to be a module homomorphism. Precisely, if R is a ring and M, N are R-modules. A mapping $f: M \to N$ is called an R-module homomorphism if it follows:

(i) $f(x+y) = f(x) + f(y)$, for all $x, y \in M$.
(ii) $f(ax) = af(x) + bf(y)$."

Definition 4.2: "A homomorphism $f: M \to N$ is called an epimorphism in case it is surjective. That is, if f maps the elements of module M onto N . An injective R-homomorphism $f: M \to N$ is called an R-monomorphism (that is, one-to-one). An R-homomorphism $f: M \to N$ is an R- isomorphism in case it is a bijection. Thus, two modules M and N are said to be isomorphic, denoted by $M \cong N$, if there is an R-

isomorphism $f : M \to N$. An R-homomorphism $f : M \to N$ is called an R-endomorphism. If $f : M \to N$ is bijective, then, it is called an R-automorphism".

Definition 4.3: (Kernel and Image of R-homomorphism)- "Let R be a ring. Let M and N be two R-modules, then a function $f : M \to N$ is a homomorphism. The Kernel of f denoted by $Ker(f)$ is given by $Ker(f) = \{x \in M \mid f(x) = 0_N\}$. The image of f denoted by $\mathrm{Im}(f)$ or $f(M)$ is defined as $\mathrm{Im}(f) = \{y \in N \mid y = f(x)$ for some $x \in M\}$".

V. TYPE OF MODULES

There are so many type of modules to study in vector space as, Finitely generated, Cyclic, Free, Projective, Injective, Flat, Torsionless, Simple, Semi-simple, Indecomposable, Faithful, Torsion-free, Noetherian, Artinian, Graded, Uniform module.

In this article a comparative study have been discussed between three modules which are,

- **Projective Modules:** "Projective modules are direct summands of free modules and share many of their desirable properties".
- **Injective Modules:** "Injective modules are defined dually to projective modules".
- **Flat Modules:** "A module is called flat if taking the tensor product of it with any exact sequence of R-modules preserves exactness".

A. Projective Modules

The class of projective modules extends the class of free modules, as modules with basic vectors on a ring, while maintaining some of the main properties of the free modules. Each free module is a projective module, but the opposite does not apply to certain rings, such as Dedekind rings. These are some of the properties mentioned below:

1. **Lifting Property:**"A module P is projective if and only if for every surjective module homomorphism $f : N^{2} \; M$ and every module homomorphism $g : P \to M$, there exists a homomorphism $h : P \to N$ such that $fh = g$".

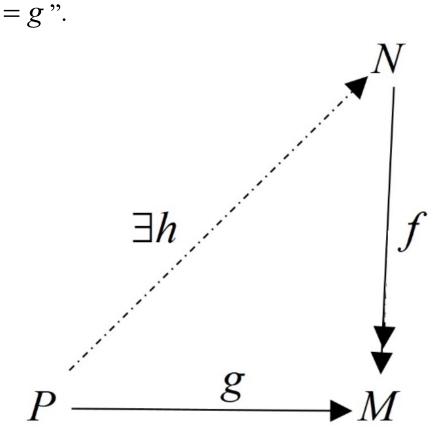

Figure 1.Explaination of Lifting Property.

2. **Split-exact sequences:** "A module P is projective if and only if every short exact sequence of modules of the form $0 \to A \to B \to P \to 0$ is a split exact sequence. That is, for every surjective module homomorphism $f : B^{2} \; P$ there exists a section map, that is, a module homomorphism $h : P \to B$ *s.t.* $fh = id_P$. In that case, $h(P)$ is a direct summand of B, h is an isomorphism from P to $h(P)$, and hf is a projection on the summand $h(P)$".

3. **Direct summands of free modules:** "A module P is projective if and only if there is another module Q such that the direct sum of P and Q is a free module".

Theorem 5.1: A free module is projective.

Proof: "Suppose that R-module F is free. Let $\alpha : K \to M$ be an R-epiomorphism and $\beta : F \to M$ be any R-homomorphism. Since F is free, it has a basis. Let S be a basis for F. $\alpha(K) = M$ and for each $s \in S$, there exist $y_s \in K$ such that $\alpha(y_s) = \beta(s)$. Define $\beta' : F \to K$ by $\beta'(s) = y_s$. Then $\alpha\beta' = \beta \Rightarrow F$ is projective".

Theorem 5.2: Every direct summand of a projective module over R is projective.

Proof: "Let $M \oplus N$ be a projective module with M, a direct summand of $M \oplus N$. Let $\pi : M \oplus N \to M$ be a projection map, $\beta : A \to B$ be an epimorphism, $\delta : M \to B$ be any R-homomorphism; where $i : M \to M \oplus N$ is an inclusion map. $\delta \circ \pi : M \oplus N \to B, \delta \circ \pi \circ i = (\delta \circ \pi) \circ i = \delta \circ (\pi \circ i) = \delta$. Then,
$\delta \circ \pi = \beta \circ \gamma \Rightarrow \delta \circ \pi \circ i = \beta \circ \gamma \circ i = \delta \Rightarrow \delta$ is a lifting to $\gamma \circ i \Rightarrow M$ is projective".

B. Injective Modules

An injective module is a module Q that shares certain desirable properties with the Z-module Q of all rational numbers. Specifically, if Q is a submodule of some other module, then it is already a direct summand of that module; also, given a submodule of a module *Y*, then any module homomorphism from this submodule to Q can be extended to a homomorphism from all of Y to Q. This concept is dual to that of projective modules.

A left module Q over the ring R is injective if it satisfies one (and therefore all) of the following equivalent conditions:

1. If Q is a submodule of some other left R-module M, then there exists another submodule K of M such that *M* is the internal direct sum of Q and K, i.e. $Q + K = M$ and $Q \cap K = \{0\}$.

2. Any short exact sequence $0 \to Q \to M \to K \to 0$ of left R-modules splits.

3. If X and Y are left R-modules, $f : X \to Y$ is an injective module homomorphism and $g : X \to Q$ is an arbitrary module homomorphism, then there exists a module homomorphism $h : Y \to Q$ such that $hf = g$, i.e. such that the following diagram commutes:

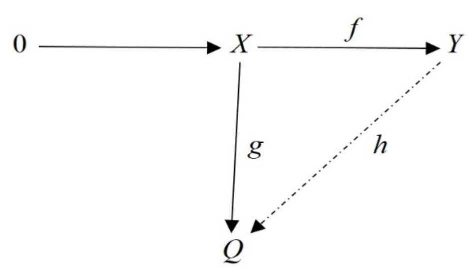

Figure 2. Explaination of Injective modules.

4. The contravariant Hom functor Hom$(-Q)$ from the category of left R-modules to the category of abelian groups is exact.

Proposition 5.1: Every unitary module A over a ring R with identtity may be embedded in an injective R-module.

Proof: "Since A is an abelian group, there is a divisible group J and a group monomorphism $f : A \to J$ by using "Every abelian group A may be embedded in a divisible abelian Group". The map $f : Hom_{\mathbb{Z}}(R,A) \to Hom_{\mathbb{Z}}(R,J)$ given on $g \in Hom_{\mathbb{Z}}(R,A)$ by $\bar{f}(g) = fg \in Hom_{\mathbb{Z}}(R,J)$ is easily seen to be an R-module monomorphism. Since every R-module homomorphism is a Z-module homomorphism, we have $Hom_R(R,A) \subset Hom_{\mathbb{Z}}(R,A)$. In fact, it is easy to see that $Hom_R(R,A)$ is an R-submodule of $Hom_{\mathbb{Z}}(R,A)$. Finally, the map $A \to Hom_R(R,A)$ given by $a \mapsto f_a$, where $f_a(r) = ra$, is an R-module monomorphism in fact it is an isomorphism. Composing these maps yeilds an R-module monomorphism $A \to Hom_R(R,A) \xrightarrow{\subset} Hom_{\mathbb{Z}}(R,A) \xrightarrow{\bar{f}} Hom_{\mathbb{Z}}(R,J)$. Since $Hom_{\mathbb{Z}}(R,J)$ is an injective R-module by using "If J is a divisible abelian group and R is a ring with identity, then $Hom_{\mathbb{Z}}(R,J)$ is an injective left R-module.", we have embedded A is an injective module".

C. Flat Modules

A flat module over a ring R is an R-module M such that taking the tensor product over R with M preserves exact sequences. A module is faithfully flat if taking the tensor product with a sequence produces an exact sequence if and only if the original sequence is exact.

"A module M over a ring R is called flat if the following condition is satisfied: for any injective map $\varnothing : K \to L$ of R-modules, the map $K \otimes_R M \to L \otimes_R M$ induced by $k \otimes m \to \varnothing(k) \otimes m$ is injective.This definition applies also if R is not necessarily commutative, and M is a left R-module and K and L right R-modules".

Theorem 5.3: Let M be an R-mdoule. If I is an ideal of R, then the map $I \otimes M \to M$ is an injection if and only if $Tor(R/I, M) = 0.$ The module M is flat if and only if this is so for every ideal I.

Proof:"Consider the exact sequence as $0 \to I \to R \to R/I \to 0.$ We obtain the exact sequence as $0 \to Tor(R/I, M) \to I \otimes M \to R \otimes M.$ Since $R \otimes M = M, Tor(R/I, M)$ is the kernel of the map from $I \otimes M$ to M. Thus the map is an injection if and only if $Tor(R/I, M) = 0.$ Now, assume that $Tor(R/I, M) = 0$ for all ideals I. Suppose that $P \to Q$ is an injection of R-modules $M \otimes P \to M \otimes Q$ is not an injection. Then there exists a non-zero element $M \otimes P$ that goes to zero. If we restrict the map to the module generated by the finitely many elements required to take $M \otimes P$ to 0, we obtain a finitely generated module for which this map is not an injection. But every finitely generated module can be decomposed into a finite chain of submodules, the successive quotients of which are cyclic modules and hence isomorphic to some R/I. Thus, $Tor(R/I, M) = 0.$ for each I implies that M is flat".

VI. CONCLUSION

As per the investigation of three modules Projective, Injective and Flat modules, A module P is created during a finite and projective way if and as long as it's a direct summand of a free module with a finite base, supposing that it's a direct summand of a free module F with a finite base, it's projective and is also a homomorphic image of a free module. Every project module is flat. the other is normally not the situation: the Abelian group Q is a Z - module is flat yet not projective. On the other hand, a flat module of finite relationship is projective. it's been demonstrated that "A module M is flat if and as long as it's an direct limit of finitely generated free modules". for the most part , the exact connection among flatness and projectivity has been built up by the fact that a M module is projective if and as long as it meets the couple of conditions: " M is Flat, It might be an association of modules from which to get is countable M satisfies a specific Mittag-Leffler type condition. Injective module is dual thereto of projective module".

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