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# ON THE NON-HOMOGENEOUS HEPTIC DIOPHANTINE EQUATION $\left(x^2-y^2\right)\left(11x^2+11y^2-20xy\right)=25\left(X^2-Y^2\right)Z^5$

S.Sriram[1], S.Kavithanandhi[2]

**Abstract- Three different methods of the non-zero solutions of non-homogeneous Heptic Diophantie equation $\left(x^2-y^2\right)\left(11x^2+11y^2-20xy\right)=25\left(X^2-Y^2\right)Z^5$ are observed. A few interesting relations between the solutions and special numbers namely pronic number, triangular number, 4dimensional figurate number, and polygonal numbers are presented.**

**Keywords: Heptic equation with five unknowns, Integral solutions, polygonal numbers, pyramidal numbers and 4D figurate number.**

2010 Mathematics Subject Classification: 11D09

*Notations Used:*

$(Pr)_a$ : Pronic number of rank a $= a(a+1)$

$(Tri)_a$ :Triangular number of rank a $= \frac{a(a+1)}{2}$

(4DF) : 4D figurate number $= \frac{a^4-a^2}{12}$

$(PenPy)_a$ :Pentagonal Pyramidal number of rank a $= \frac{a^2(a+1)}{2}$

$T_{m,n}$ : Poligonal numbers of rank n with sides m $= n\left[1+\frac{(n-1)(m-2)}{2}\right]$

$(Ptope)_a$ : Pentatope number of rank a $= \frac{a(a+1)(a+2)(a+3)}{24}$

[1]*Assistant Professor, National College (Affiliated to Bharathidasan University), Tiruchirappalli,*
[2] *Research Scholar, National College, (Affiliated to Bharathidasan University), Tiruchirappalli*

## I. INTRODUCTION

The number theory is queen of Mathematics. In particular, the Diophantine equation have a blend of attracted interesting problems. For an extensive review of variety of problems, one may refer to [1-3]. In particular heptic equations homogeneous or non-homogeneous have aroused the interest of numerous mathematicians since antiquitly [4-8]. For illlustratiobn, one may refer [9-12], for heptic equations with three, four and five unknowns. This paper concerns with the problem of determining integral solutions of the non-homogeneous heptic equation with five unknowns $\left(x^2-y^2\right)\left(11x^2+11y^2-20xy\right)=25\left(X^2-Y^2\right)Z^5$

## II. DESCRIPTION OF METHOD

Consider the Heptic Diophantine equation

$$\left(x^2-y^2\right)\left(11x^2+11y^2-20xy\right)=25\left(X^2-Y^2\right)Z^5 \quad (1)$$

We introduce the linear transformations

$$x=u+v,\quad y=u-v,\quad X=2u+v,\quad Y=2u-v \quad (2)$$

Using (2) in (1), gives to $u^2+21v^2=25Z^5$ (3)

*1.1 Pattern I*

Let us take $Z=Z(a,b)=a^2+21b^2$ (4)

where a and b are non-zero distinct integers

Take 25 as $25=\left(2+i\sqrt{21}\right)\left(2-i\sqrt{21}\right)$ (5)

Using (4) and (5) in (3) applying the factorization process, we have

$$\left(u+i\sqrt{21}v\right)=\left(2+i\sqrt{21}\right)\left(a+i\sqrt{21}b\right)^5$$

After some algebraic calculations and simplifying, we obtain

$$u=(2a^5-105a^4b-420a^3b^2+4410a^2b^3+4410ab^4-9261b^5)$$
$$v=(a^5+10a^4b-210a^3b^2-420a^2b^3+2205ab^4+882b^5)$$

Applying the value of u and v in (2) the solutions of equation (1) is given as follows

$$x(a,b)=(3a^5-95a^4b-630a^3b^2+3990a^2b^3+6615ab^4-8379b^5)$$
$$y(a,b)=(a^5-115a^4b-210a^3b^2+4830a^2b^3+2205ab^4+10143b^5)$$
$$X(a,b)=(5a^5-200a^4b-1050a^3b^2+8400a^2b^3+11025ab^4-17640b^5)$$
$$Y(a,b)=(3a^5-220a^4b-630a^3b^2+9240a^2b^3+6615ab^4-19404b^5)$$
$$Z(a,b)=a^2+21b^2$$

*Observations:*

1. $22\left[x(a,a)+y(a,a)+X(a,a)+Z(a,a)\right]+5784a\left[Z(a,a)+4DF\right]=0$

2. $Z(a,a)+2a^2$ is a nasty number.

3. $Z(1,1)-6$ is a perfect number.

4. It is observe that

i)when $b=0,\ x(a,0),y(a,0),X(a,0),Y(a,0)\ and\ Z(a,0)$ are positive for all $a>0$.

ii )when $a=0, x(0,b), y(0,b), X(0,b), Y(0,b)$ are all negative and $Z(0,b)$ is positive for all $b>0$.

5. $x(a,0)+y(a,0)+X(a,0)+Y(a,0)+12aZ(a,0)=24a\ (Tri)_{a^2}$

6. $x(0,b)-y(0,b)-X(0,b)+Y(0,b)=0$

7. $x(a,1)-3y(a,1)-22050=500(Tri)_{a^2}-5375(Hex)_a-5375a$

*2.2 Pattern II:*

From equation (3) we have $u^2+21v^2=25Z^5$ this can be written as

$$u^2+21v^2=25Z^5 * 1 \tag{6}$$

Taking 1 as $1=\frac{(1+i3\sqrt{21})(1-i3\sqrt{21})}{64}$

Substituting, equation (6) becomes $(u+i\sqrt{21}v)=\frac{(2+i\sqrt{21})(a+i\sqrt{21}b)^5(1+i3\sqrt{21})}{8}$

After some algebraic calculations, simplifying and equating real and imaginary parts we have

$$u=\frac{1}{8}(-61a^5-735a^4b+12810a^3b^2+30870a^2b^3-134505ab^4-64827b^5)$$

$$v=\frac{1}{8}(7a^5-305a^4b-1470a^3b^2+12810a^2b^3+15435ab^4-26901b^5)$$

For integer values replace $a\ by\ 2a\ and\ b\ by\ 2b$

$$u=4(-61a^5-735a^4b+12810a^3b^2+30870a^2b^3-134505ab^4-64827b^5)$$

$$v=4(7a^5-305a^4b-1470a^3b^2+12810a^2b^3+15435ab^4-26901b^5)$$

Applying the value of u and v in (2) the solutions of equation (1) is given as follows

$$x(a,b)=4(-54a^5-1040a^4b+11340a^3b^2+43680a^2b^3-119070ab^4-91728b^5)$$

$$y(a,b)=4(-68a^5-430a^4b+14280a^3b^2+18060a^2b^3-149940ab^4-37926b^5)$$

$$X(a,b) = -460a^5 - 7100a^4b + 96600a^3b^2 + 298200a^2b^3 - 1014300ab^4 - 626812b^5$$
$$Y(a,b) = -516a^5 - 4660a^4b + 108360a^3b^2 + 195720a^2b^3 - 1137780ab^4 - 411604b^5$$
$$Z(a,b) = 4(a^2 + 21b^2)$$

*Observation:*

1. The following forms a nasty numbers:

   i) $3\left[Z(a,a) - 16a^2\right]$

   ii) $Z(a,a) + 64a^2$

   iii) $-x(1,0)$

   iv) $6Z(a,0)$

   v) $6\left[x(1,0) + y(1,0) + Z(a,0) + 488\right]$

2. $x(a,a) - y(a,a) = X(a,a) - Y(a,a)$

3. It is observe that $x(a,a), y(a,a), X(a,a), Y(a,a)$ are all negative integer except $Z(a,a)$ for all $a > 0$

4. $X(a,a) - 2x(a,a) + 552a = 276a^3(Deca)_a + 1656(Tri)_{a^2} - 276(Oct)_a$

5. $Y(0,b) - 3y(0,b) - 32631b = 87016\left(b^2(PenPy)_b - (Tri)_{b^2}\right) + 10877(Deca)_b$

6. $x(1,0) + y(1,0) = 2\left[X(1,0) + Y(1,0)\right]$

7. $x(1,0) + y(1,0) + Z(a,0) + 488$ is a perfect square.

8. $x(1,0) + y(1,0) + Z(a,0) + 488 + 4a = (\Pr o)_a$

9. $x(a,a) - y(a,a) = X(a,a) - Y(a,a) = 40704(4DF) + 6784(Tri)a - 3392a$

*2.3 Pattern III*

From equation (3) we have $u^2 + 21v^2 = 25Z^5$ this can be written as

$$u^2 + 21v^2 = 25Z^5 * 1$$

taking 1 as
$$1 = \frac{\left(10 + i\sqrt{21}\right)\left(10 - i\sqrt{21}\right)}{121} \quad (7)$$

substituting for 1, equation (6) becomes

$$\left(u+i\sqrt{21}v\right)=\frac{\left(2+i\sqrt{21}\right)\left(a+i\sqrt{21}b\right)^{5}\left(10+i\sqrt{21}\right)}{11}$$

After some algebraic calculations, simplifying and equating real and imaginary parts we have

$$u=\frac{1}{11}(-a^{5}-1260a^{4}b+210a^{3}b^{2}+52920a^{2}b^{3}-2205ab^{4}-111132b^{5})$$
$$v=\frac{1}{11}\left(12a^{5}-5a^{4}b-2520a^{3}b^{2}+210a^{2}b^{3}+26460ab^{4}-441b^{5}\right)$$

To find integer solution replace 'a' by '11a' and 'b' by 11b':

$$u=11^{4}(-a^{5}-1260a^{4}b+210a^{3}b^{2}+52920a^{2}b^{3}-2205ab^{4}-111132b^{5})$$
$$v=11^{4}\left(12a^{5}-5a^{4}b-2520a^{3}b^{2}+210a^{2}b^{3}+26460ab^{4}-441b^{5}\right)$$

The solutions of (1) is given by

$$x=11^{4}(11a^{5}-1265a^{4}b-2310a^{3}b^{2}+53130a^{2}b^{3}+24255ab^{4}-111573b^{5})$$
$$y=11^{4}\left(-13a^{5}-1255a^{4}b+2730a^{3}b^{2}+52710a^{2}b^{3}-28665ab^{4}-110691b^{5}\right)$$
$$X=11^{4}(14a^{5}-2525a^{4}b-2940a^{3}b^{2}+106050a^{2}b^{3}+30870ab^{4}-222705b^{5})$$
$$Y=11^{4}\left(-10a^{5}-2515a^{4}b+2100a^{3}b^{2}+105630a^{2}b^{3}-22050ab^{4}-221823b^{5}\right)$$
$$Z=11\left(a^{2}+21b^{2}\right)$$

*Observation:*

1.The following forms a nasty numbers:

i) $Z\left(a,a\right)+242a^{2}$

ii) $Z\left(1,1\right)+242$

2. It is observe that all $x\left(a,a\right),y\left(a,a\right),X\left(a,a\right),Y\left(a,a\right)$ are negative and $Z\left(a,a\right)$ is positive for all $a>0$.

3. $Z\left(a,a\right)+2662a=2662\left(\Pr o\right)_{a}$

4. $y\left(a,a\right)-X\left(a,a\right)-11^{4}\left(4539\right)a=11^{4}\left[1513a^{3}\left(Deca\right)_{a}+9078\left(Tri\right)_{a^{2}}-4539\left(Tri\right)_{a}\right]$

5. $13x\left(a,0\right)+11y\left(a,0\right)=0$

6. $X\left(1,0\right)+Y\left(1,0\right)=4$ (bi-quadratic number).

## III. CONCLUSION

In this work, we have observed various process of determining infinitely a lot of non-zero different integer values to the non-homogeneous Heptic Diophantine equation $\left(x^2-y^2\right)\left(11x^2+11y^2-20xy\right)=25\left(X^2-Y^2\right)Z^5$ . One may try to find non-negative solutions of the above equations together with their similar observations.

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